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# ON THE STABILITY OF THE SOLUTIONS OF LINEAR STOCHASTIC INTEGRO-DIFFERENTIAL EQUATIONS ENCOUNTERED IN ELASTICITY AND VISCOELASTICITY PROBLEMS†

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The asymptotic stability, almost sure also in the mean square, of a viscoelastic system subjected to a load in the form of a random stationary broadband ergodic process is investigated. The behaviour of this system is described by integro-differential equations with stochastic parameters. The stability is considered with respect to the perturbation of the initial conditions. The governing relation is taken in an integral form with a creep (or relaxation) kernel of convolution type, which satisfies the condition of limited creep of the material. Using the fundamental solution of the corresponding deterministic integro-differential equation and its maximum Lyapunov exponent, the sufficient condition for stability of the zero solution of the initial equation or, which is the same thing, the equilibrium position of the viscoelastic system, is obtained. © 2000 Elsevier Science Ltd. All rights reserved.

Integro-differential equations are encountered both in problems of viscoelasticity and in other areas of science, where it is necessary to take into account aftereffect or delay (for example, in control theory, biology, ecology, medicine, etc. [1–7]). When describing the behaviour of elastic systems, the internal friction of the material is usually taken into account using the Voight model, although it is well known that, even in systems with a finite number of degrees of freedom, greater than unity, it leads to incorrect results, since, for the majority of materials, the internal friction is, in fact, independent of or, at least, only slightly dependent on the rate of vibrations over a fairly wide frequency band. In this sense, a model of the material which possesses hereditary properties [8, 9], which also leads to integro-differential equations, is preferable.

The problem of the stability of a viscoelastic rod, for the material of which the relaxation kernel is taken in the form of an exponential function or their sum, was solved in [10–12]. The rod is subject to a longitudinal force in the form of a stationary random process, represented by white noise. The condition for asymptotic stability with respect to the mathematical expectation and in the mean square was obtained in [11], and also the sufficient conditions for almost sure stability [10, 12].

When the load takes the form of an arbitrary stationary process, the sufficient conditions for almost sure and the mean square stability for distributed viscoelastic systems were obtained in [13].

However, an exponential-type kernel does not enable the internal damping of the material to be described adequately [9], and also the creep of many materials is described by kernels that are more complex than exponential (or degenerate). In such a case, when investigating the stability of systems it is not possible to replace the integro-differential equations by differential equations. When solving the problem of the stability of viscoelastic structural components, the material of which is subject to ageing [1–3], the external loads were assumed to be white noise. In a similar problem, the sufficient conditions for stability in the mean square were obtained for non-conservative systems in [14]. The sufficient condition for almost sure stability for a viscoelastic system under an arbitrary steady load and arbitrary relaxation kernel of the material was obtained in [12] using Lyapunov's direct method.

It should be noted that the use of Lyapunov's direct method involves choosing a suitable Lyapunov functional, which, in the case of integro-differential equations, involves overcoming such difficulties that the procedure for constructing such functionals can be compared with art [15].

Below we consider a method of investigating the stability of the zero solution of a linear integro-differential equation based on the use of the fundamental solution of an auxiliary deterministic equation.

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## 1. FORMULATION OF THE PROBLEM

We will assume that the relation between the stresses $\sigma(t)$ and the strains $\varepsilon(t)$ in the uniaxial stressed state has the form

$$\sigma = E(1-\mathbf{R})\varepsilon, \quad \mathbf{R}\varepsilon \equiv \int_0^t R(t-\tau)\varepsilon(\tau)d\tau$$

where $E$ = const is the modulus of elasticity of the material, $R(t-\tau)$ is the relaxation kernel of the material and $t$ is the time.

In the case of an isotropic material the motion of a viscoelastic system, subject to a parametric load is described by an equation which can be represented in operator form as follows:

$$\frac{\partial^2\mathbf{u}}{\partial t^2} + 2\varepsilon\frac{\partial\mathbf{u}}{\partial t} + (1-\mathbf{R})\mathbf{B}\mathbf{u} - \mathbf{C}\mathbf{u} = 0 \tag{1.1}$$

Here $\mathbf{u}(\mathbf{x}, t)$ is the displacement of the system and $\mathbf{x}$ is the spatial coordinate vector. For a fixed time $t$ the function $\mathbf{u}$ can be considered as an element of Hilbert space $H$, while the operators $\mathbf{B}$ and $\mathbf{C}$ are linear operators from $H$ into $H$.

The solution of Eq. (1.1) must satisfy the initial conditions

$$\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}), \quad \partial\mathbf{u}(\mathbf{x},t)/\partial t\big|_{t=0} = \mathbf{v}_0(\mathbf{x})$$

The terms $2\varepsilon\partial\mathbf{u}/\partial t$, $\mathbf{Bu}$, $\mathbf{Cu}$ take into account the external damping ($\varepsilon$ is a characteristic of the external friction), the stiffness of the elastic system and the action of the parametric loads, respectively.

Consider the related homogeneous problem described by the equation

$$\mathbf{Bu} = \omega^2\mathbf{u}$$

Its eigenfunctions $\varphi_1, \varphi_2, \ldots$ have the meaning of the natural forms of vibrations of the corresponding elastic system. The eigenvalues $\omega_1^2, \omega_2^2, \ldots$ are equal to the squares of the natural frequencies of this system and are related to the eigenfunctions by the Rayleigh relations

$$\omega_i^2 = (\mathbf{B}\varphi_i, \varphi_i)/(\varphi_i, \varphi_i)$$

The forms of the natural vibrations are pairwise orthogonal, so that the following equations hold

$$(\varphi_i, \varphi_j) = (\mathbf{B}\varphi_i, \varphi_i) = 0, \quad (i \neq j)$$

If $\varphi_i(\mathbf{x})$ are orthonormalized functions, $\mathbf{u}(\mathbf{x}, t)$ can be expanded in a Fourier series in these functions

$$\mathbf{u}(\mathbf{x},t) = \sum_{i=1}^{\infty} f_i(t)\varphi_i(\mathbf{x}) \tag{1.2}$$

We will further assume that the functions $\varphi_i(\mathbf{x})$ are simultaneously eigenfunctions for the operator $C$.

A similar situation is encountered fairly often [16], for example, when considering a rod of constant cross-section, hinged at the ends and subject to a longitudinal force applied at its ends, a rectangular plate of constant thickness, hinged along all the edges and subject to a uniformly distributed load acting in its plane and orthogonal to the edges, a circular cylindrical shell or cylindrical panel, rectangular in plan, hinged along the edges and subject to a uniformly distributed load, acting at the level of the middle surface of the shell and directed along the generatrix, etc.

From Eq. (1.1) we then obtain an equation for the generalized displacements $f_i(t)$

$$\ddot{f}_i + 2\varepsilon\dot{f}_i + \omega_i^2(1-\mathbf{R}-\alpha_i)f_i = 0 \tag{1.3}$$

where $\alpha_i(t)$ is a dimensionless function characterizing the parametric load. Henceforth the function $\alpha_i(t)$ will be assumed to be a stationary random ergodic process. The dot denotes a derivative with respect to time $t$.

The functions $f_i(t)$ must satisfy the initial conditions

$$f_i(0) = f_{i0}, \quad df_i(t)/dt\,|_{t=0} = v_{i0}$$

where

$$f_{i0} = \int_V \mathbf{u}_0(\mathbf{x})\varphi_i(\mathbf{x})dV, \quad v_{i0} = \int_V \mathbf{v}_0(\mathbf{x})\varphi_i(\mathbf{x})dV$$

and $V$ is the volume of the system.

## 2. THE STABILITY OF A VISCOELASTIC SYSTEM

We will introduce the norm in the space of the functions $\mathbf{u}(\mathbf{x}, t)$

$$\| \mathbf{u}(\mathbf{x},t) \|^2 = \int_V \mathbf{u}^2(\mathbf{x},t)dV$$

We will call the equilibrium position of the system $\mathbf{u}(\mathbf{x}, t) \equiv 0$ stable in the mean square with respect to a perturbation of the initial conditions if for any small positive number $\Delta$, as small as desired, there is a positive number $\delta(\Delta)$ such that, from the condition $\langle \|\mathbf{u}(0)\|^2 \rangle < \delta$, which holds for the initial instant of time $t = 0$, we obtain the inequality $\langle \|\mathbf{u}(t)\|^2 \rangle < \Delta$, that is satisfied at any instant of time $t > 0$.

Here and henceforth the angle brackets denote the operation of mathematical expectation.

The equilibrium position of the system is said to be asymptotically stable in the mean square if the previous condition is satisfied and, in addition, we obtain a $\delta > 0$ such that when $\langle \|\mathbf{u}(0)\|^2 \rangle < \delta$

$$\lim_{t\to\infty} \langle \| \mathbf{u}(t) \|^2 \rangle = 0$$

Bearing in mind expansion (1.2) and taking into account the fact that the functions $\varphi_i(\mathbf{x})$ are orthonormal, the expression for the second-order moment of the norm of the displacements can be written as follows:

$$\langle \| \mathbf{u}(t) \|^2 \rangle = \left\langle \sum_{i=1}^{\infty} f_i^2(t) \right\rangle = \sum_{i=1}^{\infty} \left\langle f_i^2(t) \right\rangle \tag{2.1}$$

Hence, to solve the problem of the stability of the viscoelastic system in question we need to obtain an estimate of the second moment of the generalized displacements $f_i(t)$.

## 3. SOLUTION OF THE INTEGRO-DIFFERENTIAL EQUATION

We will represent the stationary random process $\alpha_i(t)$ in the form of the sum

$$\alpha_i(t) = \alpha_{i0} + \alpha_i^\circ(t), \quad \alpha_{i0} = \langle \alpha_i(t) \rangle = \text{const}$$

where $\alpha_i(t)$ is a random fluctuation of the characteristics of the parametric load, $\langle \alpha_i^\circ(t) \rangle \equiv 0$.

We will rewrite Eq. (1.1) as follows:

$$\ddot{f}_i + 2\varepsilon \dot{f}_i + \omega_i^2(1 - \mathbf{R} - \alpha_{i0})f_i = \omega_i^2 \alpha_i^\circ f_i \tag{3.1}$$

We will first consider Eq. (3.1) with zero right-hand side.

We will use a Laplace transformation to solve it [17]. The transform of the function $f_i(t)$ is then given by the expression

$$\Phi_i^*(s) = s\Phi_i(s)f_{i0} + \Phi_i(s)v_{i0} \tag{3.2}$$

Here

$$\Phi_i(s) = \{s^2 + 2\varepsilon s + \omega_i^2[1 - \alpha_{i0} - \Gamma(s)]\}^{-1} \tag{3.3}$$

$\Gamma(s)$ is the transform of the relaxation kernel $R(t - \tau)$ and $s$ is a complex quantity.

We will denote the original of the transformation $\Phi_i(s)$ by $F_i(t)$. We know [17], that the original of the function $s\Phi_i(s)$ is the derivative of the function $F_i(t)$ ($F_i(0) = 0$).

As a result, integro-differential equation (3.1) can be replaced by an equivalent integral equation with kernel $F_i(t - \tau)$

$$f_i(t) = \dot{F}_i(t) f_{i0} + F_i(t) v_{i0} + \omega_i^2 \int_0^t F_i(t-\tau) \overset{\circ}{\alpha}_i(\tau) f_i(\tau) d\tau \tag{3.4}$$

We will further assume that the parameters $\varepsilon$ and $\alpha_{i0}$ are such that the zeroth solution of Eq. (3.1) with zero right-hand side is asymptotically stable. This means that the functions $F_i(t)$ and $\dot{F}_i(t)$ are bounded for any finite value of the time and tend asymptotically to zero as $t \to \infty$.

We will denote the maximum Lyapunov exponent of the solution of Eq. (3.1) with zero right-hand side by $-\lambda_i$ ($\lambda_i > 0$) ($\lambda_i$ is the characteristic number [18]) and represent the functions $F_i(t)$ and $\dot{F}_i(t)$ in the form

$$F_i(t) = \overset{\circ}{F}_i(t) e^{-\lambda_i t}, \quad \dot{F}_i(t) = F_i^*(t) e^{-\lambda_i t}$$

where the functions $\overset{\circ}{F}_i(t)$ and $F_i^*(t)$ are bounded in any finite time interval, and the maximum Lyapunov exponent for these is equal to zero.

In Eq. (3.4) we will introduce, instead of the function $f_i(t)$, a new required variable

$$y_i(t) = e^{\lambda_i t} f_i(t) \tag{3.5}$$

which is the solution of the equation obtained after substituting (3.5) into (3.4)

$$y_i(t) = \overset{\circ}{F}_i(t) f_{i0} + F_i^*(t) v_{i0} + \omega_i^2 \int_0^t F_i^*(t-\tau) \overset{\circ}{\alpha}_i(\tau) y_i(\tau) d\tau \tag{3.6}$$

From Eq. (3.6) we obtain the inequality

$$| y_i(t) | \leq G_i(t) + \omega_i^2 \int_0^t H_i(\tau) | y_i(\tau) | d\tau \tag{3.7}$$

$$C_i(t) = | \overset{\circ}{F}_i(t) f_{i0} + F_i^*(t) v_{i0} |, \quad H_i(\tau) = \eta_{i\max} \alpha_i^+(\tau) + \eta_{i\min} \alpha_i^-(\tau)$$

$$\eta_{i\max} = \sup_{\tau \in [0,t]} F_i^*(t-\tau), \quad \eta_{i\min} = \inf_{\tau \in [0,t]} F_i^*(t-\tau)$$

where $\alpha_i^+(\tau)$ and $\alpha_i^-(\tau)$ are functions of $\overset{\circ}{\alpha}_i(\tau)$ having non-negative and non-positive values, respectively.

Taking into account the boundedness of the functions $\overset{\circ}{F}_i(\tau)$ and $F_i^*(\tau)$ we have

$$G_i(t) \leq C_i, \quad C_i = \text{const}$$

and on the basis of the Gronwall–Bellman lemma [19] and inequality (3.7) we obtain an estimate of the function $|y_i(t)|$, and using it we also obtain an estimate of the absolute value of the generalized variable $|y_i(t)|$

$$| f_i(t) | \leq C_i \exp\{-\lambda_i t + \omega_i^2 \int_0^t H_i(\tau) d\tau\} \tag{3.8}$$

Since the stationary random process $\overset{\circ}{\alpha}_i(\tau)$ is centred and ergodic, we have

$$\langle \overset{\circ}{\alpha}_i(\tau) \rangle = \lim_{t \to \infty} \frac{1}{t} \int_0^t \overset{\circ}{\alpha}_i(\tau) d\tau = \lim_{t \to \infty} \frac{1}{t} \left[ \int_0^t \alpha_i^+(\tau) d\tau + \int_0^t \alpha_i^-(\tau) d\tau \right] = 0$$

Hence it follows that

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t \alpha_i^+(\tau) d\tau = -\lim_{t \to \infty} \frac{1}{t} \int_0^t \alpha_i^-(\tau) d\tau = \frac{1}{2} \lim_{t \to \infty} \int_0^t | \overset{\circ}{\alpha}_i(\tau) | d\tau = \frac{1}{2} \langle | \overset{\circ}{\alpha}_i | \rangle$$

As a result, as $t \to \infty$, we obtain from inequality (3.8)

$$| f_i(t) | \leq C_i \exp\{[-\lambda_i + \omega_i^2 (\eta_{i\max} - \eta_{i\min}) \langle | \overset{\circ}{\alpha}_i | \rangle / 2] t\} \tag{3.9}$$

It is obvious that the estimate of the mathematical expectation $\langle f_i^2(t) \rangle$ is identical with the estimate of $\overset{\circ}{f}_i(\tau)$.

Hence, we can assert that the viscoelastic system considered will be asymptotically stable in the mean square if, for each number $i$, the following inequality is satisfied

$$\left\langle |\overset{\circ}{\alpha}_i|\right\rangle < 2\lambda_i/[\omega_i^2(\eta_{i\max} - \eta_{i\min})] \tag{3.10}$$

It can be seen from inequality (3.9) that the condition obtained is simultaneously asymptotic almost sure stability condition of the equilibrium position of a viscoelastic system [13].

## 4. THE STABILITY OF A VISCOELASTIC ROD

Consider a rectilinear rod of constant cross-section, hinged at the ends and acted upon by a longitudinal force $P(t)$. Equation (1.1) in this case can be written as follows:

$$\frac{\partial^2 u}{\partial t^2} + 2\varepsilon\frac{\partial u}{\partial t} + \frac{EI}{m}(1-\mathbf{R})\frac{\partial^4 u}{\partial x^4} + \frac{P}{m}\frac{\partial^2 u}{\partial x^2} = 0$$

Here $u$ is the deflection of the rod, $x$ is a coordinate measured along the rod axis from one of its ends, $EI$ is the flexural stiffness of the rod and $m$ is its mass per unit length.

The boundary conditions at $x = 0$ and $x = l$ (where $l$ is the rod length) have the form $u = \partial^2 u/\partial x^2 = 0$.

The functions $\varphi_i(x)$, the natural frequencies $\omega_i$, the quantities $\alpha_{i0}$ and the functions $\overset{\circ}{\alpha}_{i0}$ are defined by the expressions

$$\varphi_i(x) = \sqrt{\frac{2}{l}}\sin\frac{i\pi}{l}x,\quad \omega_i^2 = \frac{i^4\pi^4 EI}{ml^4},\quad \alpha_i = \frac{Pl^2}{i^2\pi^2 EI},\quad \alpha_{i0} = \frac{\alpha_{10}}{i^2},\quad \overset{\circ}{\alpha}_i(t) = \frac{\overset{\circ}{\alpha}_1(t)}{i^2}$$

The exponential relaxation kernel. Suppose that

$$R(t-\tau) = \chi L e^{-\chi(t-\tau)},\quad \chi - \text{const},\quad L = \int_0^\infty R(\tau)d\tau = \text{const}$$

The quantity $L$ represents a measure of the material relaxation.

The transform $\Phi_i(s)$ of the fundamental solution $F_i(t)$ of Eq. (3.2) in this case has the form

$$\Phi_i(s) = \{s^2 + 2\varepsilon s + \omega_i^2[1 - \alpha_{i0} - \chi L/(s+\chi)]\}^{-1}$$

The expression in square brackets vanishes at points corresponding to the roots of the cubic polynomial. Assuming the quantities $\varepsilon$, $\chi$ ($\varepsilon, \chi \ll \omega_i$), which represent the external damping and the relaxation time, to be small compared with unity (more accurately, compared with the difference $(1 - \alpha_{i0})$, which is assumed to be positive), the roots of the cubic equation in the first approximation can be taken to be [13]

$$s_1 \approx -\chi(1-\alpha_{i0} - L)/(1-\alpha_{i0}),\quad s_{2,3} = -a \pm \sqrt{-1}\,\omega_i b$$

$$a \approx \varepsilon + \frac{\chi L}{2(1-\alpha_{i0})},\quad b \approx \sqrt{1-\alpha_{i0}-\varepsilon^2} + \frac{\varepsilon\chi L}{2(1-\alpha_{i0})\sqrt{1-\alpha_{i0}-\varepsilon^2}} \approx \sqrt{1-\alpha_{i0}}$$

Finally

$$F_i(t) = A_i e^{s_1 t} + B_1 e^{-at}\cos\omega_i bt + \frac{C_i - aB_i}{\omega_i b}e^{-at}\sin\omega_i bt$$

$$A_i = \frac{\chi L}{(1-\alpha_{i0})g} = -B_i,\ C_i = 1 - (2a + s_1)A_i,\quad g = (2a+s_1)s_1 + a^2 + \omega_i^2 b^2$$

Bearing in mind that the quantities $\varepsilon$ and $\chi$ are small, we can put

$$g \approx \omega_i^2 b^2 = \omega_i^2(1-\alpha_{i0})$$

$$F_i(t) \approx \frac{\chi L}{\omega_i^2(1-\alpha_{i0})^2}(e^{s_1 t} - e^{-at}\cos\omega_i bt) + \frac{1}{\omega_i b}e^{-at}\sin\omega_i bt$$

Further we must choose the greatest of the quantities $s_1$ and $-a$, which will also be the maximum Lyapunov exponent.

We will assume that $|s_1| < a$; then $\lambda_i = s_i$. In this case

$$F_i^*(t-\tau) \approx \frac{\chi L}{\omega_i^2(1-\alpha_{i0})^2}[1-e^{(-s_1-a)(t-\tau)}\cos\omega_i\sqrt{1-\alpha_{i0}}t]+$$
$$+\frac{1}{\omega_i\sqrt{1-\alpha_{i0}}}e^{(-s_1-a)(t-\tau)}\sin\omega_i\sqrt{1-\alpha_{i0}}t$$

To find the extremal values of $F_i^*(t)$ we can take the derivative and find $\eta_{i\max}$ and $\eta_{i\min}$ from the condition for it be zero. However, taking into account the fact that the ratio $\chi/\omega_i$ is small compared with unity, without much error we can assume

$$\begin{cases}\eta_{i\max}\\ \eta_{i\min}\end{cases} \approx \pm\frac{1}{\omega_i\sqrt{1-\alpha_{i0}}}+\frac{\chi L}{\omega_i^2(1-\alpha_{i0})^2} \tag{4.1}$$

Hence, it follows from inequality (3.11) that

$$\left\langle|\overset{\circ}{\alpha}_i|\right\rangle < \frac{\chi}{\omega_i}\frac{1-\alpha_{i0}-L}{\sqrt{1-\alpha_{i0}}} \tag{4.2}$$

If the maximum Lyapunov exponent is equal to the real part of the complex conjugate roots $s_2$ and $s_3$, we have

$$F_i^*(t-\tau) = \frac{\chi L}{\omega_i^2(1-\alpha_{i0})^2}[e^{(s_1+a)t}-\cos\omega_i bt]+\frac{1}{\omega_i b}\sin\omega_i bt$$

In this case also the quantities $\eta_{i\max}$ and $\eta_{i\min}$ can be taken in the form (4.1). Then, to estimate the quantities $\langle|\overset{\circ}{\alpha}_{i0}|\rangle$ we have the relation

$$\left\langle|\overset{\circ}{\alpha}_i|\right\rangle < [\varepsilon+\frac{\chi L}{2(1-\alpha_{i0})}]\frac{\sqrt{1-\alpha_{i0}}}{\omega_i} \tag{4.3}$$

We will compare this result with that obtained using Lyapunov's direct method, first replacing the integro-differential equation for the generalized displacement $f_i(t)$ by a system of three first-order ordinary differential equations [13]

$$\left\langle|\overset{\circ}{\alpha}_i|\right\rangle < \begin{cases}\sqrt{2}\frac{\chi}{\omega_i}\frac{1-\alpha_{i0}-L}{\sqrt{1-\alpha_{i0}}}\\ \sqrt{2}[\varepsilon+\frac{\chi L}{2(1-\alpha_{i0})}]\frac{\sqrt{1-\alpha_{i0}}}{\omega_i}\end{cases} \tag{4.4}$$

Hence, an estimate of $\langle|\overset{\circ}{\alpha}_{i0}|\rangle$ in the form (4.2) and (4.3) turns out to be more rigorous than estimate (4.4). This result might have been expected since the specific features of the exponential kernel of the material relation was in no way taken into account.

A refinement of the values of $\eta_{i\max}$ and $\eta_{i\min}$ does not lead to any appreciable changes in the results, which is confirmed by the data in Table 1, where we show values of the estimate of $\langle|\overset{\circ}{\alpha}_{i0}|\rangle_*$, obtained numerically from (3.10) and the similar values of $\langle|\overset{\circ}{\alpha}_{i0}|\rangle^*$ obtained from (4.4).

The values of $s_1$ and $-a$, shown in Table 1, were obtained by numerical solution of the cubic equation

$$\{s^2+[2\varepsilon s+\omega_i^2(1-\alpha_{i0})]\}(s+\chi)-\omega_i^2\chi L=0$$

It can be shown that for small values of $\varepsilon$, $\chi$ and $L$ they are practically identical with the similar values obtained for $s_1$ and $a$ written above.

The results shown in Table 1 illustrate the effect of different parameters on the value of the estimate of $\langle|\overset{\circ}{\alpha}_i|\rangle$. On the one hand, as might have been expected, as the value of the mathematical expectation of the longitudinal force increases there is a reduction in the estimate of $\langle|\overset{\circ}{\alpha}_{i0}|\rangle_*$. On the other hand, it is noteworthy that an increase in the measure of relaxation for fixed values of the parameters $\varepsilon$ and $\chi$ does not lead to any appreciable change in the same estimate, at least for small values of $\alpha_0$. It is also interesting to note that a change in each of the parameters $\varepsilon$ and $\chi$ separately, keeping the other

Table 1

| $\varepsilon$ | $\chi$ | $L$ | $\alpha_0$ | $-s_1\times10^5$ | $a\times10^5$ | $\langle\lvert\overset{\circ}{\alpha}\rvert\rangle_*\times10^5$ | $\langle\lvert\overset{\circ}{\alpha}\rvert\rangle^*\times10^5$ |
|---|---|---|---|---|---|---|---|
| 0.01 | 0.04 | 0.1 | 0 | 3600 | 1200 | 1075 | 1697 |
| | | | 0.25 | 3467 | 1266 | 970 | 1551 |
| | | | 0.50 | 3201 | 1400 | 861 | 1400 |
| | | | 0.75 | 2401 | 1800 | 786 | 1273 |
| | | | 0.90 | 0 | 3000 | 0 | 0 |
| 0.04 | 0.01 | 0.1 | 0 | 900 | 4050 | 993 | 1273 |
| | | | 0.25 | 867 | 4067 | 843 | 1061 |
| | | | 0.50 | 800 | 4100 | 655 | 800 |
| | | | 0.75 | 599 | 4200 | 374 | 424 |
| | | | 0.90 | 0 | 4500 | 0 | 0 |
| 0.04 | 0.04 | 0.1 | 0 | 3600 | 4200 | 3680 | 5091 |
| | | | 0.25 | 3466 | 4267 | 3103 | 4246 |
| | | | 0.50 | 3198 | 4401 | 2406 | 3200 |
| | | | 0.75 | 2391 | 4804 | 1417 | 1697 |
| | | | 0.90 | 0 | 6000 | 0 | 0 |
| 0.04 | 0.04 | 0.01 | 0 | 3960 | 4020 | 3548 | 5600 |
| | | | 0.25 | 3947 | 4027 | 3016 | 4834 |
| | | | 0.50 | 3920 | 4040 | 2407 | 3920 |
| | | | 0.75 | 3839 | 4048 | 1607 | 2885 |
| | | | 0.90 | 3594 | 4203 | 937 | 1610 |
| 0.04 | 0.04 | 0.05 | 0 | 3800 | 4100 | 3642 | 5734 |
| | | | 0.25 | 3733 | 4133 | 3127 | 4572 |
| | | | 0.50 | 3599 | 4201 | 2548 | 3600 |
| | | | 0.75 | 3195 | 4402 | 1740 | 2263 |
| | | | 0.90 | 1976 | 5012 | 872 | 894 |
| 0.04 | 0.04 | 0.005 | 0 | 3980 | 4010 | 3537 | 5629 |
| | | | 0.25 | 3974 | 4013 | 3002 | 4866 |
| | | | 0.50 | 3960 | 4020 | 2389 | 3960 |
| | | | 0.75 | 3920 | 4040 | 1581 | 2772 |
| | | | 0.90 | 3797 | 4101 | 889 | 1699 |
| 0.02 | 0.02 | 0.00707 | 0 | 1986 | 2007 | 1888 | 2808 |
| | | | 0.25 | 1981 | 2009 | 1621 | ·2426 |
| | | | 0.50 | 1972 | 2014 | 1303 | 1972 |
| | | | 0.75 | 1944 | 2028 | 902 | 1374 |
| | | | 0.90 | 1858 | 2071 | 546 | 831 |

parameters unchanged, although it also leads to a change in the value of $\langle\lvert\overset{\circ}{\alpha}_i\rvert\rangle_*$, this change is not very large (for small $\alpha_0$). Nevertheless, a simultaneous change in the same parameters, again keeping the other parameters fixed, leads to a considerable change in the estimate of $\langle\lvert\overset{\circ}{\alpha}_i\rvert\rangle_*$.

These assertions confirm the quite complex relationship between the parameters of the external damping and the viscosity of the material and the critical values of the parameter $\langle\lvert\overset{\circ}{\alpha}_i\rvert\rangle_*$.

It follows from (4.2) an d (4.3) that if they are satisfied when $i = 1$, they will be all the more satisfied when $i > 1$. On the basis of this we can assert that the rod will be asymptotically stable in the mean square (and almost sure) for an arbitrary form of the perturbing initial conditions if it is asymptotically stable for perturbations having the form of a single sinusoidal half-wave.

*A relaxation kernel with a weak singularity.* Consider the following relaxation kernel of a material

$$R(t-\tau) = \frac{B}{\Gamma(\nu)}\frac{e^{-a(t-\tau)}}{(t-\tau)^{\nu}}$$

where $B$, $a$ and $\nu$ are constants, $0 < \nu \leqslant 1$. We will confine ourselves to analysing the stability of a rod when the perturbation of the initial conditions takes the form of a single half-wave of a sinusoid and hence the subscript will henceforth be omitted.

A measure of the relaxation for such a kernel is

$$L = \int_0^\infty R(\tau)d\tau = \frac{B}{a^\nu}$$

The transform of the function $R(t)$ has the form

$$\Gamma(s) = B/(s + a)^\nu$$

The transform of the function $F(t)$ can be written as follows:

$$\Phi(s) = [(s+a)^2 + 2(\varepsilon - a)(s+a) + D - B\omega^2/(s+a)^\nu]^{-1}$$
$$D = \omega^2(1-\alpha_0) + a(a-2\varepsilon)$$

from which it follows that

$$F(t) = e^{-at}\psi(t) \tag{4.5}$$

In turn, the transform of the function $\psi(t)$ is

$$\Psi^*(s) = [s^2 + 2(\varepsilon - a)s + D - B\omega^2/s^\nu]^{-1} \tag{4.6}$$

If $\nu$ is a rational fraction, the transform $\Psi(s)$ can be represented as a rational fraction, which in turn can be written in the form of the sum of simple fractions.
Further, as an example we will consider the special case when $\nu = 1/2$ and $\omega = 1$.
It can be shown that the transform (4.6) has the original [17]

$$\psi(t) = \frac{1}{2\sqrt{\pi}t^{3/2}}\int_0^\infty \tau \exp\left(-\frac{\tau^2}{4t}\right)\phi(\tau)d\tau$$

In turn, the function $\phi(\tau)$ is the original of the transform

$$\Phi^*(s) = s/[s^5 + 2(\varepsilon - a)s^3 + Ds - B] \tag{4.7}$$

We will assume that the roots of the equation

$$s^5 + 2(\varepsilon - a)s^3 + Ds - B = 0 \tag{4.8}$$

are a single real root $s_1$ and two pairs of complex-conjugate roots

$$s_{2,3} = \lambda_1 \pm i\theta_1, \quad s_{4,5} = \lambda_2 \pm i\theta_2, \qquad i = \sqrt{-1}$$

Then

$$\varphi(t) = A_1 e^{-s_1 t} + e^{-\lambda_1 t}(A_2 \cos\theta_1 t + A_3 \sin\theta_1 t) + e^{-\lambda_2 t}(A_4 \cos\theta_2 t + A_5 \sin\theta_2 t)$$

Here $A_1, \ldots, A_5$ are constants.
As a result, the fundamental solution $F(t)$ can be represented in the form of the sum of functions

$$F_1(t) = A_1 e^{-at}\left[\frac{2}{\sqrt{\pi t}} - s_1 e^{s_1^2 t}\operatorname{erfc}(s_1\sqrt{t})\right]$$

$$F_{2j}(t) = A_{2j}\left[\frac{1}{\sqrt{\pi t}}e^{-at} - K_j^+(t)\right], \quad F_{2j+t}(t) = iA_{2j+1}K_j^-(t)$$

$$K_j^\pm(t) = \frac{1}{2}e^{(-a\pm\lambda_j^2 \mp\theta_j^2)t}\Big\{(\lambda_i - i\theta_j)e^{\mp i2\lambda_j\theta_j t}\operatorname{erfc}[(\lambda_j - i\theta_j)\sqrt{t}] \pm$$
$$\pm(\lambda_j + i\theta_j)e^{\pm i2\lambda_i\theta_j t}\operatorname{erfc}[(\lambda_j + i\theta_j)\sqrt{t})]\Big\}$$

Table 2

| $\varepsilon = a$ | $L$ | $B$ | $\alpha_0$ | $-a_1 \times 10^5$ | $-a_2 \times 10^5$ | $-a_3 \times 10^5$ | $\langle\vert\alpha_0\vert\rangle \times 10^5$ |
|---|---|---|---|---|---|---|---|
| 0.04 | 0.05 | 0.01 | 0 | 3990 | 4353 | 3645 | 3720 |
| | | | 0.25 | 3982 | 4438 | 3559 | 3157 |
| | | | 0.50 | 3960 | 4594 | 3401 | 2504 |
| | | | 0.75 | 3838 | 4995 | 2985 | 1643 |
| 0.04 | 0.005 | 0.001 | 0 | 4000 | 4035 | 3965 | 3982 |
| | | | 0.25 | 4000 | 4044 | 3956 | 3432 |
| | | | 0.50 | 4000 | 4060 | 3940 | 2795 |
| | | | 0.75 | 3998 | 4100 | 3899 | 1965 |
| 0.02 | 0.00707 | 0.001 | 0 | 1995 | 2250 | 1750 | 1773 |
| | | | 0.25 | 1991 | 2310 | 1689 | 1488 |
| | | | 0.50 | 1980 | 2419 | 1578 | 1149 |
| | | | 0.75 | 1920 | 2703 | 1287 | 688 |

The maximum Lyapunov exponent of the function $F(t)$ will be equal to the greatest of the expressions

$$a_1 = -a + s_1^2, \quad a_2 = -a + \lambda_j^2 - \theta_j^2, \quad a_3 = -a - \lambda_j^2 + \theta_j^2$$

It is then easy to obtain an expression for the function $F^*(t)$.

The forms of the function $F(t)$ and for the other combinations of the roots of Eq. (4.8) are obvious.

As an example we will assume that the quantities $\varepsilon$, $a$ and $B$ are fairly small. The roots of Eq. (4.8) in the first approximation then turn out to be (when $\varepsilon = a$)

$$s_1 \approx B/D = B/(1 - \alpha_0 - a^2)$$

$$s_{2,3} \approx \lambda_1 \pm i\theta, \quad s_{4,5} \approx \lambda_2 \pm i\theta$$

$$\lambda_1 = \sqrt{D}/2 - s_1/4, \quad \lambda_2 = -\sqrt{\sqrt{D}/2} - s_1 4, \quad \theta = \sqrt{\sqrt{D}/2}$$

The maximum and minimum values of the function $F^*(t)$ are easier to obtain numerically and hence we obtain an estimate of the quantity $\langle|\overset{\circ}{\alpha}|\rangle$.

The results of calculations of $\langle|\overset{\circ}{\alpha}|\rangle$ for some values of the parameters $\varepsilon = a$, $B$ and $\alpha_0$ are shown in Table 2.

A comparison of the data in Table 2 with the similar data in Table 1 shows that for the same values of the parameters $L$, $\varepsilon$, $\varkappa$ and $a$ the values of the estimate $\langle|\overset{\circ}{\alpha}_i|\rangle$ in Table 2 are practically identical with the similar values of $\langle|\overset{\circ}{\alpha}_i|\rangle_*$ for all values of the parameter $\alpha_0$ in Table 1. The difference increases as the mathematical expectation of the longitudinal force increases. This indicates that, from the point of view of system stability, a consideration of the weak singularity in the relaxation kernel of the material cannot have any appreciable influence on the values of the critical parameter, whereas the presence of this singularity considerably complicates the solution of the problem.

The examples considered confirm that the proposed method is an effective means of investigating the stability of the zeroth solution of linear stochastic integro-differential equations (the equilibrium positions of viscoelastic systems), which enables one to obtain an estimate of the critical value of $\langle|\alpha_0|\rangle$ for fairly arbitrary relaxation kernels of the material.

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